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Solitary magnon excitations in a one-dimensional antiferromagnet with Dzyaloshinsky–Moriya interactions

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Abstract
We investigate solitary excitations in a model of a one-dimensional antiferromagnet including a single-ion anisotropy and a Dzyaloshinsky–Moriya antisymmetric exchange interaction term. We employ the Holstein–Primakoff transformation, the coherent state ansatz and the time variational principle. We obtain two partial differential equations of motion by using the method of multiple scales and applying perturbation theory. By so doing, we show that the motion of the coherent amplitude must satisfy the nonlinear Schrödinger equation. We give the single-soliton solution.

1. Introduction

The study of nonlinear excitations in one-dimensional (1D) magnetic systems has attracted a great deal of theoretical and experimental interest in the last five decades and continued to be the subject of numerous investigations [1, 2]. It is well known that the ferromagnetic (FM) and antiferromagnetic (AFM) chain compounds such as $CsNiF_3$ and TMMC [$(CH_3)_4NMnCl_3$] are systems exhibiting soliton-like excitations [2]. Several theoretical and experimental methods for studying nonlinear excitations of 1D and 2D magnets have been proposed [2–9]. One of the most interesting methods is that of coherent state treatment. In the spin–coherent state representation, the exact nonlinear equation of motion for the system is obtained directly with spin operators without any approximations as regards the Hamiltonian [10]. Other coherent state treatments use a truncated Holstein–Primakoff transformation of the spin operator $S_j^{\pm}$ and further approximate the Hamiltonian in the boson operators [11, 12]. Huang *et al* investigated solitary excitations in the alternating FM Heisenberg chain and in the biquadratic anisotropic spin chain [13–15], using Glauber's coherent state representation. Nguenang *et al* investigated solitary excitations including localized and delocalized structures using the coherent state representation [16–18]. They set up a hierarchy of bound states induced by a wave train, and they observed the localization phenomenon realized by the presence of an intrinsic localized mode in a 1D Heisenberg ferromagnet [17]. Liu *et al* studied the solitary excitations in order-parameter-preserving AFM in systems such as $CsFeBr_3$, using a coherent state ansatz [19–22].

For most of the above-mentioned studies on the corresponding materials, the model Hamiltonian includes a single-ion anisotropy in addition to the exchange energy and a Zeeman term. However, materials such as $BaCu_2M_2O_7$ (M $=$ Si, Ge) [23, 24], La_2CuO_4 [25], Yb_4As_3 [26], YVO_3–$SrVO_3$ [27] and $CuGeO_3$ [28–30] cannot be described with a simple Hamiltonian. Experimental investigations of high field neutron scattering measurements and electron paramagnetic resonance investigations have shown that the Dzyaloshinsky–Moriya (DM) interaction plays an important role for these materials [26, 31, 32]. Despite their importance, there has been little work on the nonlinear dynamics of a 1D Hamiltonian including the DM interaction term. Zaspel used a classical approach for the investigation of topological solitons [33]. The quantum soliton in Cu benzoate materials [34] was studied through a different approach. Reference [34] uses a transformation and approximations that lead to a Hamiltonian which provides a grasp of the essentials of quantum fluctuations [34]. A renormalization group method and a quantum field theoretic approach led to a sine–Gordon system and a gap was obtained. This model system can be used to describe topological solitons such as kinks. As in the case of $CsNiF_3$, a study of the solitary excitations in the large density limit would be interesting in such systems. For the study of nonlinear excitations in the AFM chain with DM antisymmetric exchange interactions [35, 36], the coherent state ansatz should now be applied in a straightforward way. In this paper we study a model Hamiltonian including DM antisymmetric exchange interactions. We employ the Holstein–Primakoff transformation and the coherent state ansatz and investigate nontopological solitary wave solutions. We predict the possibility for a transition from nontopological solitary wave excitations to a topological solitary wave solution in the system.

This paper is organized as follows. In section 2, we employ the Holstein–Primakoff transformation and the coherent state ansatz, and obtain two partial differential equations of motion from the Hamiltonian. Then we use the long wavelength approximation and the method of multiple scales. Hence, we reduce these equations to envelope function equations and find that the amplitude function satisfies a nonlinear Schrödinger equation. In section 3, we use the inverse scattering transformation to obtain a single-soliton solution. The last section is devoted to the conclusions.

2. Model Hamiltonian and equations of motion

The Hamiltonian of the AFM system with DM antisymmetric exchange interactions under consideration is given by

$$H = J\sum_l \left(\vec{S}_l^A\vec{S}_l^B + \vec{S}_{l+1}^A\vec{S}_l^B\right) + D\vec{z}\sum_l\left(\vec{S}_l^A\Lambda\vec{S}_l^B + \vec{S}_{l+1}^A\Lambda\vec{S}_l^B\right) + A\left[\sum_l\left(S_{lz}^A\right)^2 + \sum_l\left(S_{lz}^B\right)^2\right] \quad (1)$$

where $J > 0$ is the exchange parameter, and $D > 0$ and $A > 0$ measure the strength of the DM interaction and the single-ion anisotropy, respectively. The Landé factor is g, and μ_B is the Bohr magneton. The symbol Λ stands for the vector product. We consider only nearestneighbour interactions. The AFM is divided into two interpenetrating sublattices A and B. We use the Holstein–Primakoff transformation [37]

$$S_l^{A+} = \left(2S - a_l^\dagger a_l\right)^{1/2} a_l; \qquad S_l^{A-} = a_l^\dagger \left(2S - a_l^\dagger a_l\right)^{1/2}; \qquad S_{lZ}^A = S - a_l^\dagger a_l \tag{2}$$

$$S_l^{B+} = b_l^\dagger \left(2S - b_l^\dagger b_l\right)^{1/2}; \qquad S_l^{B-} = \left(2S - b_l^\dagger b_l\right)^{1/2} b_l; \quad S_{lZ}^B = -S + b_l^\dagger b_l \tag{3}$$

where the a_l ($a_l^\dagger$) and b_l ($b_l^\dagger$) are annihilation (creation) operators associated with the A and B sublattices and S is the magnitude of the spin operator. With this transformation, the Hamiltonian turns out to be bosonized. For the usual spin wave approximation we use only quadratic terms in the bosonized Hamiltonian. We will look for localized nonlinear spin excitations. In order to calculate such large amplitude collective modes we use the coherent states method [15–20, 38]. We define α_l, β_l as the coherent state amplitudes satisfying the relations $a_l|\psi(t)\rangle = \alpha_l|\psi(t)\rangle$, $b_l|\Psi(t)\rangle = \beta_l|\Psi(t)\rangle$, and $|0\rangle$ is the vacuum state of the boson system. Then using the time dependent variational principle as in [19–22, 39], we obtain the equations of motion for the coherent state amplitudes α_l and β_l as

$$\begin{aligned} \mathrm{i}\hbar\frac{\mathrm{d}\alpha_l}{\mathrm{d}t} = {} & [JS + A(1-2S)]\alpha_l + (J - \mathrm{i}D)S(\beta_l^* + \beta_{l-1}^*) + 2A|\alpha_l|^2\alpha_l \\ & - \tfrac{1}{4}(J - D\mathrm{i})(|\beta_l|^2\beta_l^* + |\beta_{l-1}|^2\beta_{l-1}^*) - \tfrac{1}{2}(J - \mathrm{i}D)|\alpha_l|^2(\beta_l^* + \beta_{l-1}^*) \\ & - \tfrac{1}{4}(J + \mathrm{i}D)(\beta_l + \beta_{l-1})\alpha_l^2 - J\alpha_l(|\beta_l|^2 + |\beta_{l-1}|^2) \\ & - \frac{1}{32S}[(J - \mathrm{i}D)\alpha_l^2|\beta_l|^2\beta_l + (J + \mathrm{i}D)\alpha_l^2\beta_l + (J - \mathrm{i}D)|\beta_l|^2\beta_l^* \\ & + (J - \mathrm{i}D)\alpha_l^2|\alpha_l|^2\beta_l] - \frac{1}{32S}[3(J - \mathrm{i}D)|\alpha_l|^4\beta_l^* + 2(J - \mathrm{i}D)|\alpha_l|^2\beta_l^* \\ & + 4(J - \mathrm{i}D)|\alpha_l|^2|\beta_l|^2\beta_l^* + (J - \mathrm{i}D)|\beta_l|^4\beta_l^*] \end{aligned} \tag{4a}$$

$$\begin{aligned} \mathrm{i}\hbar\frac{\mathrm{d}\beta_l}{\mathrm{d}t} = {} & [JS + A(1-2S)]\beta_l + (J - \mathrm{i}D)S(\alpha_l^* + \alpha_{l+1}^*) + 2A|\beta_l|^2\beta_l \\ & - \tfrac{1}{4}(J - \mathrm{i}D)(|\alpha_l|^2\alpha_l^* + |\alpha_{l+1}|^2\alpha_{l+1}^*) - \tfrac{1}{2}(J - \mathrm{i}D)|\beta_l|^2(\alpha_l^* + \alpha_{l+1}^*) \\ & - \tfrac{1}{4}(J + iD)(\alpha_l + \alpha_{l-1})\beta_l^2 - J\beta_l(|\alpha_l|^2 + |\alpha_{l+1}|^2) \\ & - \frac{1}{32S}[(J - \mathrm{i}D)\beta_l^2|\alpha_l|^2\alpha_l + (J + \mathrm{i}D)\beta_l^2\alpha_l + (J - \mathrm{i}D)|\alpha_l|^2\alpha_l^* \\ & + (J + \mathrm{i}D)\beta_l^2|\beta_l|^2\alpha_l] - \frac{1}{32S}[3(J - \mathrm{i}D)|\beta_l|^4\alpha_l^* + 2(J - \mathrm{i}D)|\beta_l|^2\alpha_l^* \\ & + 4(J - \mathrm{i}D)|\alpha_l|^2|\beta_l|^2\alpha_l^* + (J - \mathrm{i}D)|\alpha_l|^4\alpha_l^*] \end{aligned} \tag{4b}$$

where α_l^*, β_l^* are complex conjugates of α_l, β_l. Linearizing equations (4a) and (4b), we obtain the linear dispersion relation

$$\hbar\omega_\pm = A\,(1 - 2S) \pm 2S\left\{\left(J^2 + D^2\right)\left(\sin^2(ka)\right)\right\}^{1/2}, \tag{5}$$

where the symbol ($\pm$) labels the two branches: ω_+ denotes the 'optic branch' and ω_- the 'acoustic branch', by analogy with the phonon case. The dispersion curve is shown in figure 1, which shows a gap at the edge of the Brillouin zone, i.e. for $k = k_\mathrm{B} = 0, \frac{\pi}{a}$. The 'optical branch' ω_+ has a maximum value ω_max and a minimum value ω_min, while the 'acoustic branch' ω_- also has corresponding maximum and minimum values $\omega_{-\,\mathrm{max}}$ and $\omega_{-\,\mathrm{min}}$. It is also important to mention that unlike in the case of dynamics of the lattice when $k = 0$, $\omega_{-\,\mathrm{max}} \neq 0$, the excitation energy does not vanish in the long wavelength approximation.

We stress that in the low temperature excitations, the introduction of a coherent state helps in projecting out the unphysical boson states provided that no essential information is lost in

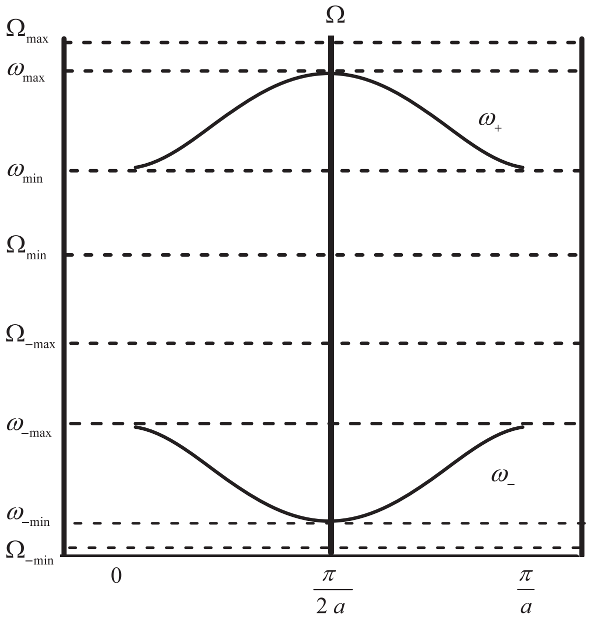

Figure 1. The spectrum of the frequencies shows how the solitary excitation frequencies enter into the frequency gap of the linear dispersion relation. This is the proof for the nonlinear localized vibration modes in the antiferromagnetic chain with the DM exchange interaction.

this treatment. In this respect, working with the boson operators should also lead to keeping essential features of our one-dimensional AFM spin chain with the DM interaction.

Equations (4*a*) and (4*b*) are difficult to solve since they are nonlinear, discrete and coupled. We assume the characteristic wavelength λ_0 of the solitons to be larger than the lattice constant $2a$, i.e. $\lambda_0 \gg 2a$, then we can take the long wavelength approximation, i.e.

$$\alpha_l(t) \to \psi_1(x,\,t); \qquad \alpha_{l\pm1}(t) \to \psi_1(t) \pm \eta\psi_{1x} + \frac{1}{2!}\eta^2\psi_{1xx} \pm \frac{1}{3!}\eta^3\psi_{1xxx} + \mathrm{O}\left(\eta^4\right) \tag{6}$$

$$\beta_l(t) \to \psi_2(x,\,t); \qquad \beta_{l\pm1}(t) \to \psi_2(t) \pm \eta\psi_{2x} + \frac{1}{2!}\eta^2\psi_{2xx} \pm \frac{1}{3!}\eta^3\psi_{2xxx} + \mathrm{O}\left(\eta^4\right) \tag{7}$$

where $\eta = 2a/\lambda_0$ is a small dimensionless parameter. Substituting equations (6) and (7) into equations (4*a*) and (4*b*), and retaining only terms up to $\mathrm{O}(\eta^2)$ and neglecting the constant terms whose presence does not modify the form of the solution, the equations of motion become

$$\begin{aligned} \mathrm{i}\hbar\frac{\mathrm{d}\psi_1}{\mathrm{d}t} &= [JS + A\,(1-2S)]\,\psi_1 + SU\left(2\psi_2 - \eta\psi_{2x} + \frac{1}{2}\eta^2\psi_{2xx}\right)^* - \frac{1}{2}U|\psi_2|^2\psi_2^* \\ &\quad - \tfrac{1}{2}U\,|\psi_1|^2\left(2\psi_2 - \eta\psi_{2x} + \tfrac{1}{2}\eta^2\psi_{2xx}\right)^* - \tfrac{1}{4}U^*\psi_2^*\left(2\psi_2 - \eta\psi_{2x} + \tfrac{1}{2}\eta^2\psi_{2xx}\right) \\ &\quad - 2J\,|\psi_2|^2\,\psi_1 + 2A\,|\psi_1|^2\,\psi_1 - \frac{1}{32}U^*\psi_1^2\psi_2 - \frac{1}{32S}U\,|\psi_2|^2\,\psi_2^* \\ &\quad - \frac{1}{16S}U\,|\psi_1|^2\,\psi_2^* \end{aligned} \tag{8}$$

$$\begin{aligned} \mathrm{i}\hbar\frac{\mathrm{d}\psi_2}{\mathrm{d}t} &= [2JS + A(1-2S)]\,\psi_2 + SU\left(2\psi_1 - \eta\psi_{1x} + \frac{1}{2}\eta^2\psi_{1xx}\right)^* - \frac{1}{2}U\,|\psi_1|^2\,\psi_1^* \\ &\quad - \tfrac{1}{2}U\,|\psi_2|^2\left(2\psi_1 - \eta\psi_{1x} + \tfrac{1}{2}\eta^2\psi_{1xx}\right)^* - \tfrac{1}{4}U^*\psi_2^*\left(2\psi_1 - \eta\psi_{1x} + \tfrac{1}{2}\eta^2\psi_{1xx}\right) \end{aligned}$$

$$- 2J\,|\psi_1|^2\,\psi_2 + 2A\,|\psi_2|^2\,\psi_2 - \frac{1}{32}U^*\psi_2^2\psi_1 - \frac{1}{32S}U\,|\psi_1|^2\,\psi_1^* - \frac{1}{16S}U\,|\psi_2|^2\,\psi_1^* \tag{9}$$

where U stands for a complex number given by $U = J - \mathrm{i}D$ and U^* is its complex conjugate.

To solve equations (8) and (9), we use the method of multiple scales [40, 41] which will help in reducing these equations to a system of nonlinear equations which allows us to find the slowly varying components of functions ψ_1 and ψ_2 [36]. To this end we introduce length scales $x_j = \mu^j x$ and timescales $t_j = \mu^j t$ ($\mu \ll 1$, $j = 1, 2, 3\ldots$) which are considered independent. In this consideration, the first spatial and temporal derivatives and the expansions of quantities ψ_1, ψ_2 are

$$\frac{\partial}{\partial t} = \mu\frac{\partial}{\partial t_1} + \mu^2\frac{\partial}{\partial t_2} + \mu^3\frac{\partial}{\partial t_3} + \cdots, \tag{10a}$$

$$\frac{\partial}{\partial x} = \mu\frac{\partial}{\partial x_1} + \mu^2\frac{\partial}{\partial x_2} + \mu^3\frac{\partial}{\partial x_3} + \cdots. \tag{10b}$$

$$\psi_1 = \mu\psi_1^{(1)} + \mu^2\psi_1^{(2)} + \mu^3\psi_1^{(3)}\ldots \tag{11a}$$

$$\psi_2 = \mu\psi_2^{(1)} + \mu^2\psi_2^{(2)} + \mu^3\psi_2^{(3)}\ldots. \tag{11b}$$

Here, $\psi_1^{(j)}$ and $\psi_2^{(j)}$ depend on x_j and t_j. Substituting equations (10) and (11) into equations (8) and (9) and collecting terms with equal power of μ, the following equations for $\psi_1^{(j)}$, $\psi_2^{(j)}$ are obtained:

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - \omega_k\right)\psi_k^{(j)} - G_k\psi_m^{(j)^*} = \phi_k^{(j)} \tag{12}$$

$$\omega_k = \delta_1 JS + A(1-2S) \tag{13}$$

$$G_k = SU\left(2 + \delta_2\eta\frac{\partial}{\partial x} + \frac{1}{2}\eta^2\frac{\partial^2}{\partial x^2}\right) \tag{14}$$

$$\phi_k^{(1)} = 0 \tag{15}$$

$$\phi_k^{(1)} = -\mathrm{i}\hbar\frac{\partial(\psi_k^{(1)})}{\partial t_1} + \delta_2 SU\eta\frac{\partial\psi_m^{(1)*}}{\partial x_1} + SU\eta^2\frac{\partial^2(\psi_m^{(1)*})}{\partial x\partial x_1} \tag{16}$$

$$\begin{aligned}\phi_k^{(3)} = {}& -\mathrm{i}\hbar\frac{\partial(\psi_k^{(1)})}{\partial t_2} - \mathrm{i}\hbar\frac{\partial(\psi_k^{(2)})}{\partial t_1} + \delta_2 SU\eta\left(\frac{\partial\psi_l^{(1)*}}{\partial x_2} + \frac{\partial\psi_l^{(2)*}}{\partial x_1}\right)\\ &+ \frac{SU\eta^2}{2}\left(\frac{2\partial^2\psi_l^{(1)}}{\partial x\partial x_2} + \frac{\partial^2\psi_l^{(1)}}{\partial x_1^2} + 2\frac{\partial^2\psi_l^{(2)}}{\partial x\partial x_1}\right)^* - \frac{1}{2}U\left|\psi_m^{(1)}\right|\psi_m^{(1)^*}\\ &- \tfrac{1}{2}U|\psi_k^1|^2\left(2\psi_m^1 + \eta\psi_{mx}^{(1)} + \tfrac{1}{2}\eta^2\psi_{mxx}^{(1)}\right)^*\\ &- \tfrac{1}{4}U^*\psi_k^{(1)^2}\left(2\psi_m^1 + \eta\psi_{mx}^{(1)} + \tfrac{1}{2}\eta^2\psi_{mxx}^{(1)}\right)\\ &- \tfrac{1}{32}\Big[U^*(\psi_k^{(1)})^2\psi_m^1 + 3U|\psi_k^{(1)}|^2\psi_m^{(1)*} + 64J|\psi_m^{(1)}|^2\psi_k^{(1)}\\ &+ 64A|\psi_k^{(1)}|^2\psi_k^{(1)}\Big] \qquad \delta_1 = k, \quad \delta_2 = (-1)^k.\end{aligned} \tag{17}$$

The indices $k, m = 1, 2$ with $k \neq m$ run over the two sublattices A and B respectively, and the indices $j = 1, 2, 3, \ldots$ stand for the different orders of expansion.

Before we continue, it is important to stress that, due to the absence of the magnetic field in the dispersion relation (see equation (5)), a gap is opened up in the system either by the single-anisotropy term or the Dzyaloshinsky–Moriya antisymmetric exchange term or by both. This is an unusual phenomenon because very often in antiferromagnetic chains, one of the roles of the external applied magnetic field is to open up the gap. Here it is not necessary. This result is also important in the sense that there are some materials such as $RbCoCl_3{\cdot}2H_2O$ [42] where it happens that the exchange energy parameter (J) is very weak as compared to the Dzyaloshinsky exchange constant (D). In such a case the gap is induced by the antisymmetric exchange and the anisotropy interaction.

3. Solitary excitations

We now rewrite equation (12) in the following form:

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t}-\omega_1\right)\left(\mathrm{i}\hbar\frac{\partial}{\partial t}-\omega_2\right)\psi_1^{(j)}-G_1G_2^*\psi_1^{(j)}=\left(\mathrm{i}\hbar\frac{\partial}{\partial t}-\omega_2\right)\phi_1^{(j)}-G_1\phi_2^{(j)^*} \tag{18}$$

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t}-\omega_2\right)\psi_2^{(j)}-G_2\psi_1^{(j)^*}=\phi_2^{(j)}. \tag{19}$$

For $j=1$, we obtain the equations

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t}-\omega_1\right)\left(\mathrm{i}\hbar\frac{\partial}{\partial t}-\omega_2\right)\psi_1^{(1)}-G_1G_2^*\psi_1^{(1)}=0 \tag{20}$$

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t}-\omega_2\right)\psi_2^{(1)}-G_2\psi_1^{(1)^*}=0, \tag{21}$$

which are linear wave equations with solutions

$$\psi_1^{(1)}=B\,(x_1,x_2,\ldots,t_1,t_2\ldots)\exp[\mathrm{i}\,(kx-\omega t+\alpha_0)]+\text{c.c.}, \tag{22}$$

$$\psi_2^{(1)}=\frac{-h}{\omega-\omega_2}\psi_1^{(1)}+\text{c.c.}, \tag{23}$$

$$h=SU\left(-2-\mathrm{i}\eta k+\frac{\eta^2k^2}{2}\right) \tag{24}$$

and the dispersion relation is

$$2\hbar\omega_\pm=3JS+2A\,(1-2S)\pm\left\{(JS)^2+4S^2\left(J^2+D^2\right)\left(4-\eta^2k^2\right)\right\}^{1/2} \tag{25}$$

where B is an arbitrary complex function of the slow scales to be determined. Equation (22) is the long wavelength approximation of equation (5). For $j=2$, we have

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t}-\omega_1\right)\left(\mathrm{i}\hbar\frac{\partial}{\partial t}-\omega_2\right)\psi_1^{(2)}-G_1G_2^*\psi_1^{(2)}=\left(\mathrm{i}\hbar\frac{\partial}{\partial t}-\omega_2\right)\phi_1^{(2)}-G_1\phi_2^{(2)^*} \tag{26}$$

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t}-\omega_2\right)\psi_2^{(2)}-G_2\psi_1^{(2)^*}=\phi_2^{(2)}. \tag{27}$$

Using (22) and (23), we obtain

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t}-\omega_1\right)\left(\mathrm{i}\hbar\frac{\partial}{\partial t}-\omega_2\right)\psi_1^{(2)}-G_1G_2^*\psi_1^{(2)}$$
$$=\mathrm{i}\hbar\,(\omega_1+\omega_2-2\omega)\left(\frac{\partial B}{\partial t_1}-c_{\mathrm{g}}\frac{\partial B}{\partial z_1}\right)\exp\left(\mathrm{i}\,[kx-\omega t]\right) \tag{28}$$

where $c_g = \omega'(k)$ is the group velocity of linear waves. The right-hand side of equation (28) is a secular term [40, 41]. To apply the perturbation theory, the function B must satisfy the following equation:

$$\frac{\partial B}{\partial t_1} - c_g \frac{\partial B}{\partial x_1} = 0. \tag{29}$$

From equation (29) we deduce that

$$B = B(\xi, x_2, \ldots; t_2 \ldots) \qquad \text{with } \xi = x_1 + c_g t_1. \tag{30}$$

We conclude that the waves travel with group velocity on the slow scales of space and time. The solutions of equations (26), (27) are

$$\psi_1^{(2)} = B(\xi, x_2, \ldots; t_2 \ldots) \exp[\mathrm{i}(kx - \omega t + \alpha_0)] + \text{c.c.} \tag{31}$$

$$\psi_2^{(2)} = -\frac{1}{\omega - \omega_2}\left[hB \exp\{\mathrm{i}(kx - \omega t + \alpha_0)\} - \left(\frac{\mathrm{i}\hbar c_g}{\omega - \omega_2} + SU\eta + \mathrm{i}SU\eta^2 k\right)\frac{\partial B}{\partial \xi}\right] \times \exp\{\mathrm{i}(kx - \omega t + \alpha_0)\} + \text{c.c.} \tag{32}$$

For $j = 3$, we obtain the equations

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - \omega_1\right)\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - \omega_2\right)\psi_1^{(3)} - G_1 G_2^* \psi_1^{(3)} = (\omega_1 + \omega_2 - 2\omega)\left(\mathrm{i}\hbar\frac{\partial B}{\partial t_2} + \frac{1}{2}\omega''(k)\frac{\partial^2 B}{\partial \xi^2} + \lambda |B|^2 B\right)\exp(\mathrm{i}[kx - \omega t]) \tag{33}$$

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - \omega_2\right)\psi_2^{(3)} - G_2 \psi_1^{(3)^*} = \phi_2^{(3)} \tag{34}$$

where

$$\omega''(k) = \frac{2S^2\eta^2|U|^2 + 2\hbar^2 c_g^2}{\omega_1 + \omega_2 - 2\omega}, \qquad c_g = \omega'(k), \qquad \xi = x + c_g t \tag{35}$$

$$\begin{aligned}\lambda = \frac{3}{(\omega_1 + \omega_2 - 2\omega)}\Bigg\{&-\frac{(Uh^* + U^*h)((\omega - \omega_2)^2 + (h + h^*))}{4(\omega - \omega_2)^2} \\ &+ 2(\omega - \omega_2)\left(A + JS\left(\frac{U + U^*}{2}\right)\frac{\omega - \omega_1}{\omega - \omega_2}\right)\Bigg\} - \frac{3}{\omega_1 + \omega_2 - 2\omega} \\ &\times \left\{2\frac{(h + h^*)^2}{(\omega - \omega_2)}\left(J + AS\left(\frac{U + U^*}{2}\right)\frac{\omega - \omega_1}{\omega - \omega_2}\right) - \frac{(Uh^* + U^*h)}{2}\right. \\ &\times \left.\left[\left(\frac{1}{2} + \frac{1}{32S}\right)(h + h^*)^2 + \left(\frac{1}{16S} + 1\right)\frac{(h + h^*)^2}{(\omega - \omega_2)^2} + \frac{1}{48S}\right]\right\}.\end{aligned} \tag{36}$$

Application of perturbation theory [40, 41] forces the second member of equation (33) to be zero, and then function B satisfies the equation

$$\mathrm{i}\hbar\frac{\partial B}{\partial t_2} + \frac{1}{2}\omega''(k)\frac{\partial^2 B}{\partial \xi^2} + \lambda|B|^2 B = 0. \tag{37}$$

Equation (40) is a nonlinear Schrödinger equation which is completely integrable and has an exact solution if we use the inverse scattering transformation [40, 41]. A single-soliton solution of (37) is

$$B = \left(\frac{c_0^2\omega''(k)}{\mu^2\lambda}\right)^{1/2} \sec h\left\{c_0\left[x - x_0 - \left(c_g + c_1\omega''(k)\right)t\right]\right\} \times \exp\{\mathrm{i}[(k + c_1)x - \Omega t - \varphi_0]\} \tag{38}$$

$$\Omega = \omega + c_1 c_g + \tfrac{1}{2}\left(c_1^2 - c_0^2\right)\omega''(k). \tag{39}$$

Here, the parameters c_0, c_1, x_0, φ_0 are integration constants. Equation (38) describes a wavepacket which travels with velocity $c_g + c_1\omega''(k)$. If $c_1 = 0$, then the solitary excitation frequency given in equation (38) reduces to $\Omega = \omega - \frac{1}{2}c_0^2\omega''(k)$. For $k = 0, \frac{\pi}{a}$ and $k = \frac{\pi}{2a}$ ($c_g = 0$), for the 'acoustic branch' and the 'optical branch', the solitary excitation frequency displays four values, i.e.

$$\begin{aligned}
&\Omega_{\max}\left(k = \frac{\pi}{2a}\right) = \omega_{\max} - \frac{1}{2}c_0^2\omega''_{\max} > \omega_{\max},\\
&\Omega_{-\max}(k = 0) = \omega_{-\max} - \tfrac{1}{2}c_0^2\omega''_{-\max} > \omega_{-\max}\\
&\Omega_{\min}(k = 0) = \omega_{\min} - \tfrac{1}{2}c_0^2\omega''_{\min} < \omega_{\min}\\
&\Omega_{-\min}\left(k = \frac{\pi}{2a}\right) = \omega_{-\min} - \frac{1}{2}c_0^2\omega''_{-\min} < \omega_{-\min}.
\end{aligned} \tag{40}$$

It is therefore clear that these solitary excitation frequencies enter into the gap of the linear dispersion relation in equation (5) and denote the solitary localized modes in an AFM with DM interaction such as $BaCu_2M_2O_7$ (M = Si, Ge) etc.

The configuration of the spin for the solution with a single solitary wave h is obtained:

$$\left\langle S_{lZ}^{A}\right\rangle = S - \frac{c_0^2\omega''(k)}{\mu^2\lambda}\sec h^2\left\{c_0\left[x - x_0 - \left(c_g + c_1\omega''(k)\right)t\right]\right\} \tag{41}$$

$$\left\langle S_{lZ}^{B}\right\rangle \to -S + \frac{S^3A^2c_0^2\omega''(k)(4 + k^2a^2)}{4\lambda(\omega - \omega_2)^2}\sec h^2\left\{c_0\left[x - x_0 - (c_g + c_1\omega''(k))\right]\right\}. \tag{42}$$

Equations (41) and (42) show explicitly magnon localization in our AFM. They are solitary excitations in the system studied. Needless to say, these results were obtained thanks to the Holstein–Primakoff transformation, the coherent state ansatz and the time dependent variational principle that leads to two partial differential equations. It is very difficult to solve these equations because of the nonlinearity and discreteness. But using the method of multiple scales helps with reducing these equations to an envelope function equation and then we force the amplitude function to satisfy a nonlinear Schrödinger equation (see equation (37)). This latter equation plays an important role in many nonlinear phenomena and has been widely studied. When looking at equation (37) it is realized that the coefficient $\omega''(k)$ that is defined in equation (35) plays the role of the dispersion phenomenon that a given wave may face while propagating in such a system. This is certainly attested to by its frequency dependence and hence the wavevector dependence of the linear spin wave. However, the presence of the nonlinear term with the coefficient λ would help generate solitary magnon localization in an antiferromagnet with a DM interaction term. On the other hand, we mention that the spin configuration of equation (42) appears with the possibility of a resonance denominator. This can be understood as the signature of a possible resonance in the system that would lead to a transition between the ground state and the first excited state. This would confirm the presence of the Haldane gap in such a system with a possibility for spin configurations changing from a nontopological structure to a topological shape, this being followed by the occurrence of a resonant kink. Another possibility is that this may lead to a complete delocalization of the magnetic excitation in the case of the absence of equilibrium between the dispersion phenomenon and nonlinearity in the system. Further details on this aspect will be given in a later paper.

4. Summary

In this paper we have investigated the existence of solitary excitations in a model of an AFM with DM antisymmetric exchange interactions. By introducing the Holstein–Primakoff transformation and the coherent state ansatz we obtain two partial nonlinear and coupled differential equations. We used the method of multiple scales to reduce theses equations to a nonlinear Schrödinger equation, whose solutions and properties are well known. The obtaining of the spatial configuration of the spin proved the existence of the solitary excitations in the system studied. It is important to mention that taking into account the DM term also led to finding the frequency of the solitary excitations in the gap of the linear wave as in the case of the AFM system. This gap is opened up in the system either by the single-anisotropy term or by the Dzyaloshinsky–Moriya antisymmetric exchange term or by both, unlike the case for the isotropic AFM system. Since the approach used here is semi-classical, it should be clear that the result cannot be used to deal with an $S = \frac{1}{2}$ quantum spin chain such as $CuGeO_3$. This poses the problem of the consistency of the method of coherent states for solitary excitation in the FM and AFM. In the case of the FM the consistency of the semi-classical treatment has been examined [14–16]. The nonlinear modified terms in the equation of motion of coherent amplitude are strongly constrained by a relation between the spin magnitude ε $(=S^{-1/2})$ and the characteristic length of the soliton λ_0. Skrinjar *et al* [12] showed that it is impossible to establish a direct relation between ε and λ_0. They could not avoid the inconsistency of the two-parameter theory of solitons in magnetic systems. The approach used here for the study of solitary excitations in AFM with DM interaction is consistent and systematic. We were also able to predict a gap in the AFM with DM interaction as in [34] whenever our method was different. Finding some frequencies of the solitary excitations in the gap of linear waves can be understood as a prediction of a signature of the gap soliton in such an AFM with antisymmetric exchange. Due to a challenge as regards the consistency of results, further theoretical, computational and experimental investigations are strongly encouraged, with the aim of investigating details of these soliton bearing systems, trying to establish a more quantitative picture.

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